

# Solitary waves in coupled nonlinear Schrödinger equations with spatially inhomogeneous nonlinearities

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Using Lie group theory we construct explicit solitary wave solutions of coupled nonlinear Schrödinger systems with spatially inhomogeneous nonlinearities. We present the general theory, use it to construct different families of explicit solutions and study their linear and dynamical stability.

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## I. INTRODUCTION

The nonlinear Schrödinger equation (NLS) is one of the universal models of mathematical physics arising in many different physical scenarios [1, 2]. In recent years there has been an enormous interest on the study of NLS equations with nonlinear coefficients depending either on space, time or both. The motivation comes from the applications of the model to the fields of Bose-Einstein condensates (BECs) and nonlinear optics. In the first field of application, the so-called Feshbach resonance management allows for a precise control of the atomic interactions responsible for the strength of nonlinearities. This has been experimentally implemented leading to time dependent nonlinear coefficients and would allow for the generation of space or space-time dependent nonlinearities.

Many different types of nonlinear waves have been studied theoretically, experimentally or numerically in the context of BECs (see e.g. the reviews [3, 4]). However the consideration of inhomogeneous nonlinearities has led to the prediction of many remarkable nonlinear phenomena in the last few years, either for time dependent [5–15] or space dependent nonlinear coefficients [16–37].

Although the model equations of interest are not integrable in general, there have been efforts to construct families of exact solutions of the NLS with spatially inhomogeneous coefficients. Specifically, group-theoretical methods based on Lie symmetries have been used to obtain solutions of NLS type equations in Refs. [24, 25, 31, 36, 38–40]. However, up to now the applications of the technique have been restricted to scalar systems. The purpose of this paper is to extend the method and construct exact solutions for multicomponent systems. These systems are of an enormous physical interest because of the extra degrees of freedom provided by the existence of two independent subsystems, e.g. in term of application to multicomponent BEC the model describes two hyperfine levels of an atomic BEC. Exact soliton solutions for such systems in the case of homogeneous in time and space nonlinearities have been reported in many papers (see e.g. [41–47] and references therein). Very recently in [48] the dynamics of the multicomponent BEC with spatially modulated nonlinearity was treated by means of the the variational approach. Here we will go beyond previous studies and use Lie symmetry method to construct exact bright-bright, dark-dark and dark-bright solutions of the two-component system with spatially inhomogeneous nonlinearities and investigate their linear and dynamics stability.

The paper is organized as follows. First, in Section II, we introduce the general theory of the Lie symmetry analysis for our model problem: a system of two coupled nonlinear Schrödinger equations with spatially inhomogeneous nonlinearities. In Section III, we use the method to construct explicit solutions for systems without external potentials ( $V_j(x) = 0$ ,  $j = 1, 2$ ) and study the linear and dynamical stability of the obtained solutions. In Section IV, we present some families of exact solutions for inhomogeneous coupled nonlinear Schrödinger equations with external potentials quadratic in space and propose their stability analysis. Next, in Section V, we discuss the construction of dark-bright

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soliton solutions and study their stability. Finally, in Section VI, we summarize our conclusions. The theory of the linear stability analysis for the system is presented in Appendix A.

## II. GENERAL THEORY

### A. Model to be studied

In this paper we will construct and study the stability of solutions of the following coupled spatially inhomogeneous nonlinear Schrödinger equations (CINLS)

$$i \frac{\partial \psi_1}{\partial t} = -\frac{\partial^2 \psi_1}{\partial x^2} + V_1(x) \psi_1 + (g_{11}(x) |\psi_1|^2 + g_{12}(x) |\psi_2|^2) \psi_1, \quad (1a)$$

$$i \frac{\partial \psi_2}{\partial t} = -\frac{\partial^2 \psi_2}{\partial x^2} + V_2(x) \psi_2 + (g_{21}(x) |\psi_1|^2 + g_{22}(x) |\psi_2|^2) \psi_2, \quad (1b)$$

for known real potential functions  $V_j(x)$ , and spatial modulations of the nonlinearities described by the real functions  $g_{jk}(x)$ ,  $j, k = 1, 2$ . In applications to BEC,  $\psi_1$  and  $\psi_2$  are complex wave functions describing two hyperfine levels (the extension to different atomic species is simple, corresponding to the addition of different masses) and describe the dynamics of coupled quasi-one dimensional BECs in the mean field limit with spatially inhomogeneous interactions.

In the following we will be interested in solitary wave solutions of Eqs. (1) of the form  $\psi_j(x, t) = u_j(x) e^{-i\lambda_j t}$ , where  $u_j(x)$  are real functions ( $j = 1, 2$ ). Upon substitution in Eqs. (1) we get system of stationary equations

$$\lambda_1 u_1 = -\frac{d^2 u_1}{dx^2} + V_1(x) u_1 + (g_{11}(x) u_1^2 + g_{12}(x) u_2^2) u_1, \quad (2a)$$

$$\lambda_2 u_2 = -\frac{d^2 u_2}{dx^2} + V_2(x) u_2 + (g_{21}(x) u_1^2 + g_{22}(x) u_2^2) u_2. \quad (2b)$$

The construction of families of exact solutions of Eqs. (2) is the main goal of this paper.

### B. Lie symmetry method for Eqs. (2)

In this section we will extend the method of Ref. [24] to the CINLS given by Eqs. (2). The methodology is similar although the calculations are more complicated, but we will show that it is still possible to extend the method for this more complicated situation.

It is said that a second-order differential system

$$N_j(x, u_j, u_{j,x}, u_{j,xx}) = 0, \quad (3)$$

possesses a Lie point symmetry [49] of the form

$$M = \xi(x, u_1, u_2) \frac{\partial}{\partial x} + \eta_1(x, u_1, u_2) \frac{\partial}{\partial u_1} + \eta_2(x, u_1, u_2) \frac{\partial}{\partial u_2}, \quad (4)$$

if the action of the second extension of  $M$ , i.e.  $M^{(2)}$  on  $N_j$ , gives zero whenever (3) is satisfied, i.e.

$$\begin{aligned} M^{(2)} N_j(x, u_j, u_{j,x}, u_{j,xx}) = & \left[ \xi(x, u_1, u_2) \frac{\partial}{\partial x} + \eta_1(x, u_1, u_2) \frac{\partial}{\partial u_1} + \right. \\ & \eta_2(x, u_1, u_2) \frac{\partial}{\partial u_2} + \eta_1^{(1)}(x, u_1, u_2, u_{1,x}) \frac{\partial}{\partial u_{1,x}} + \eta_2^{(1)}(x, u_1, u_2, u_{2,x}) \frac{\partial}{\partial u_{2,x}} \\ & \left. + \eta_1^{(2)}(x, u_1, u_2, u_{1,xx}) \frac{\partial}{\partial u_{1,xx}} + \eta_2^{(2)}(x, u_1, u_2, u_{2,xx}) \frac{\partial}{\partial u_{2,xx}} \right] N_j(x, u_j, u_{j,x}, u_{j,xx}) = 0 \end{aligned} \quad (5)$$

where  $u_{j,x} \equiv du_j/dx$  and  $u_{j,xx} \equiv d^2 u_j/dx^2$ , with  $j = 1, 2$ .

In our problem,  $N_j(x, u_j, u_{j,x}, u_{j,xx})$ ,  $j = 1, 2$  are given by

$$N_1(x, u_1, u_2, u_{1,xx}) = u_{1,xx} - f_1(x, u_1, u_2), \quad (6a)$$

$$N_2(x, u_1, u_2, u_{2,xx}) = u_{2,xx} - f_2(x, u_1, u_2), \quad (6b)$$

where

$$f_1(x, u_1, u_2) = [V_1(x) - \lambda_1 + g_{11}(x)u_1^2 + g_{12}(x)u_2^2] u_1, \quad (7a)$$

$$f_2(x, u_1, u_2) = [V_2(x) - \lambda_2 + g_{21}(x)u_1^2 + g_{22}(x)u_2^2] u_2, \quad (7b)$$

and the action of the operator  $M^{(2)}$  on  $N_1$  leads to

$$\xi_{u_1 u_1} = 0, \quad (8a)$$

$$\xi_{u_1 u_2} = 0, \quad (8b)$$

$$\xi_{u_2 u_2} = 0, \quad (8c)$$

$$\eta_{1, u_2 u_2} = 0, \quad (8d)$$

$$\eta_{1, u_1 u_2} = \xi_{u_2, x}, \quad (8e)$$

$$\eta_{1, u_2 x} = \xi_{u_2} f_1, \quad (8f)$$

$$\eta_{1, u_1 u_1} = 2\xi_{x u_1}, \quad (8g)$$

$$2\eta_{1, x u_1} - \xi_{xx} - 3\xi_{u_1} f_1 - f_2 \xi_{u_2} = 0, \quad (8h)$$

$$\eta_{1, xx} + (\eta_{1, u_1} - 2\xi_x) f_1 - \xi f_{1, x} - \eta_1 f_{1, u_1} - \eta_2 f_{1, u_2} + f_2 \eta_{1, u_2} = 0. \quad (8i)$$

In a similar way, we get equations for the action of  $M^{(2)}$  on  $N_2$  (we omit the details). Solving the previous equations, we find that the only Lie point symmetries of Eq. (2) are of the form

$$M = c(x) \frac{\partial}{\partial x} + m(x) u_1 \frac{\partial}{\partial u_1} + m(x) u_2 \frac{\partial}{\partial u_2}, \quad (9)$$

where

$$g_{jk}(x) = \tilde{g}_{jk}/c(x)^3, \quad (10a)$$

$$m(x) = c'(x)/2, \quad (10b)$$

$$-c(x)V_1'(x) + \frac{c'''(x)}{2} - 2c'(x)(V_1(x) - \lambda_1) = 0, \quad (10c)$$

$$V_2(x) = V_1(x) + K/c^2(x) + \lambda_2 - \lambda_1. \quad (10d)$$

Here  $\tilde{g}_{jk}$  = constants and  $K$  an arbitrary real number.

It is known [50], that the invariance of the energy is associated to the translational invariance. The generator of such a transformation is of the form  $M = \partial/\partial x$ . To use this fact, we define the transformation

$$X = h(x), \quad U_1 = v(x)u_1, \quad U_2 = v(x)u_2, \quad (11)$$

where  $h(x)$  and  $v(x)$  will be determined by requiring that a conservation law of energy type  $M = \partial/\partial X$  exists in the canonical variables. Using Eqs. (9) and (11), we get

$$h(x) = \int_0^x \frac{1}{c(s)} ds, \quad v(x) = c(x)^{-1/2}. \quad (12)$$

Rewriting Eqs. (2) in terms of  $U_j = c^{-1/2}(x)u_j$ ,  $j = 1, 2$  and  $X = \int_0^x 1/c(s)ds$  we get

$$-\frac{d^2 U_1}{dX^2} + (\tilde{g}_{11}U_1^2 + \tilde{g}_{12}U_2^2) U_1 = E_1 U_1, \quad (13a)$$

$$-\frac{d^2 U_2}{dX^2} + (\tilde{g}_{21}U_1^2 + \tilde{g}_{22}U_2^2) U_2 = E_2 U_2, \quad (13b)$$

where

$$E_1 = (\lambda_1 - V_1(x)) c(x)^2 - \frac{1}{4}c'(x)^2 + \frac{1}{2}c(x)c''(x), \quad (14a)$$

$$E_2 = (\lambda_2 - V_2(x)) c(x)^2 - \frac{1}{4}c'(x)^2 + \frac{1}{2}c(x)c''(x), \quad (14b)$$

are constants. Thus, in the new variables we obtain nothing else but system of NLS equations *without external potential and with spatially homogeneous nonlinearities*. Of course not any choice of  $V_j(x)$  and  $g_{jk}(x)$ ,  $j, k = 1, 2$ ,

leads to the existence of a Lie symmetry or an appropriate canonical transformation since the function  $c(x)$  must be sign definite for  $U_j$ ,  $j = 1, 2$  and  $X$  to be properly defined. In the specific case when all nonlinear coefficients are equal, i.e.  $\tilde{g}_{jk} = \tilde{g}$  for all  $j, k = 1, 2$ , Eqs. (13) correspond to the time independent Manakov system [41], that has been studied in detail (see, for instance [42–44]).

We can thus use the known solutions of Eqs. (14b) to construct solutions of Eqs. (2) in the same spirit as in the scalar case (see e.g. Ref. [22]). Moreover, when both  $E_1 = E_2 = E$  and the restriction

$$(\tilde{g}_{11} - \tilde{g}_{21})(\tilde{g}_{22} - \tilde{g}_{12}) > 0, \quad (15)$$

are satisfied, Eqs. (13) reduce to the scalar NLS equation

$$-\frac{d^2\Phi}{dX^2} + \sigma\Phi^3 = E\Phi, \quad (16)$$

where we have introduced the new field  $\Phi$ , related with the original fields through the relation

$$U_j = \theta_j \Phi. \quad (17)$$

Here  $\theta_1 = \sqrt{|\tilde{g}_{22} - \tilde{g}_{12}|/Q}$ , and  $\theta_2 = \sqrt{|\tilde{g}_{11} - \tilde{g}_{21}|/Q}$ , being  $Q = |\tilde{g}_{11}\tilde{g}_{22} - \tilde{g}_{12}\tilde{g}_{21}|$  and  $\sigma = \text{sgn}[(\tilde{g}_{11}\tilde{g}_{22} - \tilde{g}_{12}\tilde{g}_{21})(\tilde{g}_{jj} - \tilde{g}_{12})]$  which due to the condition (15) does not depend on  $j$ . In that situation we can use the many known solutions of the scalar equation (16) to construct solutions of the vector problem (2). Equation (16) has many known exact solutions. Remarkable examples, to be used later in this paper, are

$$\Phi = \mu \tanh\left(\frac{\mu}{\sqrt{2}}X\right), \quad E = 2\mu^2, \quad \sigma = 1, \quad (18a)$$

$$\Phi = \sqrt{2}\mu \frac{1}{\cosh(\mu X)}, \quad E = -\mu^2, \quad \sigma = -1, \quad (18b)$$

$$\Phi = \mu k \sqrt{2(1-k^2)} \frac{\text{sn}(\mu X, k)}{\text{dn}(\mu X, k)}, \quad E = \mu^2(1-2k^2), \quad \sigma = -1, \quad (18c)$$

$$\Phi = \sqrt{2}\mu \text{dn}(\mu X, k), \quad E = \mu^2(k^2-2), \quad \sigma = -1. \quad (18d)$$

### III. SYSTEMS WITHOUT EXTERNAL POTENTIAL ( $V_1(x) = V_2(x) = 0$ )

#### A. General considerations

As a first application of our ideas let us choose  $V_1(x) = V_2(x) = 0$ . By taking  $K = 0$  and  $\lambda_2 = \lambda_1 \equiv \lambda$  in (10d) we find from Eq. (10c) the equation  $c'''(x) + 4\lambda c'(x) = 0$ , whose general solutions are given by

$$c(x) = C_1 \sin \omega x + C_2 \cos \omega x + C_3 \quad (\lambda > 0), \quad (19a)$$

$$c(x) = C_1 e^{\omega x} + C_2 e^{-\omega x} + C_3 \quad (\lambda < 0), \quad (19b)$$

where  $\omega = 2\sqrt{|\lambda|}$ . It should be emphasized that the auxiliary function  $c(x)$  is a periodic function for  $\lambda > 0$  which corresponds to the continue spectrum of the linear eigenvalue problem  $d^2m/dx^2 + 4\lambda m = 0$  and for  $\lambda < 0$  (the region where the linear eigenvalue problem does not have periodic solutions) the function  $c(x)$  becomes exponentially localized. Therefore, in the following subsections we will distinguish two types of solutions of Eqs. (1): corresponding to either periodic or localized nonlinearities.

#### B. Dark-dark soliton solutions

Let us first consider the case  $\lambda > 0$ . Then, taking  $C_1 = 0$ ,  $C_2 = \alpha$ ,  $C_3 = 1$ , Eq. (19a) leads to a periodic dependence of the coefficient, e.g.  $c(x) = 1 + \alpha \cos \omega x$ , and according to Eq. (10a) the nonlinear coefficients are given by

$$g_{jk}(x) = \tilde{g}_{jk} (1 + \alpha \cos \omega x)^{-3}, \quad j, k = 1, 2. \quad (20)$$

Although the form of nonlinear coefficients seems complicated, it is easy to see that for small  $\alpha$  this nonlinearity is approximately harmonic in space  $g_{jk}(x) \simeq \tilde{g}_{jk} (1 - 3\alpha \cos \omega x)$ ,  $\alpha \ll 1$ ,  $j, k = 1, 2$ , and thus at least in that limit is of direct physical interest.

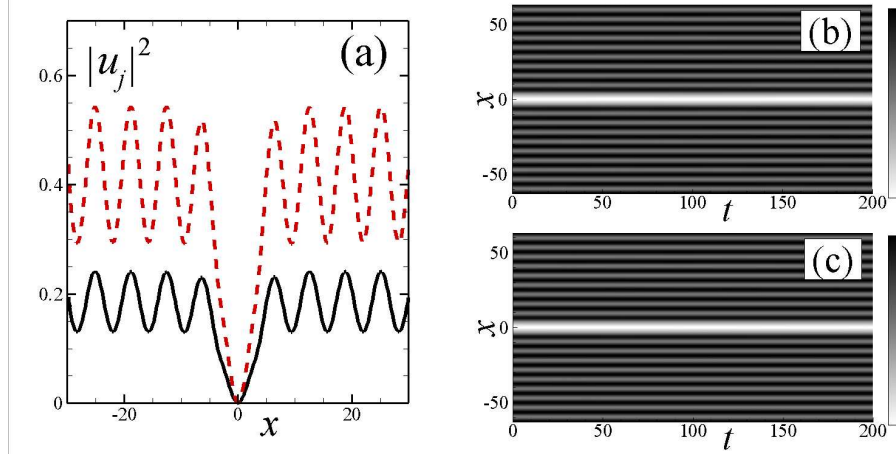


FIG. 1: Dynamics of the dark-dark solution (21) under a perturbation of the form (23). Shown are (a) the initial density profiles of the first (black solid) and second (red dashed) component. (b,c) Density plot of the time evolution of (b)  $|u_1(x,t)|^2$  (c)  $|u_2(x,t)|^2$ . Parameter values are  $g = 0.1$ ,  $g_{11} = 1$ ,  $g_{22} = 0.5$ ,  $\alpha = 0.3$ ,  $\omega = 1$ ,  $\epsilon = 0.001$ ,  $q = 0.01$ .

We can construct our canonical transformation by using Eqs. (11) and obtain

$$\tan\left(\frac{\omega}{2}\sqrt{1-\alpha^2}X(x)\right) = \sqrt{\frac{1-\alpha}{1+\alpha}} \tan\frac{\omega x}{2}. \quad (21)$$

Using any solution of Eq. (13) with  $E = E_1 = E_2 = \frac{1}{4}\omega^2(1-\alpha^2) > 0$ , where  $\lambda = \omega^2/4$ , this transformation provides solutions of Eq. (2) with  $g(x)$  given by Eq. (20). For instance, taking the nonlinear coefficients to satisfy Eq. (15) and  $\sigma = 1$  we can construct dark-dark soliton solutions of Eq. (2) of the form

$$u_j(x) = \theta_j(1 + \alpha \cos(\omega x))^{1/2} \sqrt{2}\mu \tanh(\mu X(x)), \quad (22)$$

where  $\mu = \sqrt{E} = \frac{1}{2}\omega\sqrt{1-\alpha^2}$  and  $X(x)$  is defined by Eq. (21). We want to emphasize that this is only a simple example of the many possible solutions that can be constructed in such a way.

### C. Stability of the dark-dark solution (22)

Let us consider the dynamical stability of the dark-dark soliton solution (22). Here and in all of our calculations to be presented in this paper in order to check dynamical stability we perturb the initial exact solution according to

$$u_j(x) = u_{j0}[1 + \epsilon \cos(qx \pm 1)], \quad (23)$$

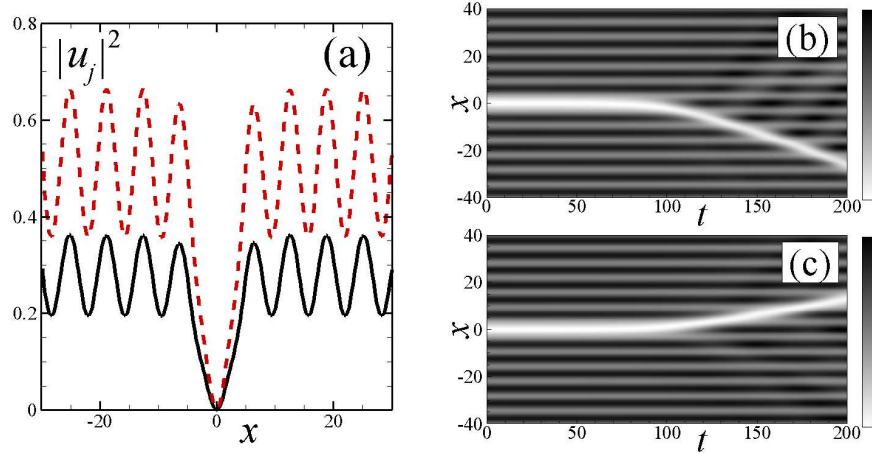
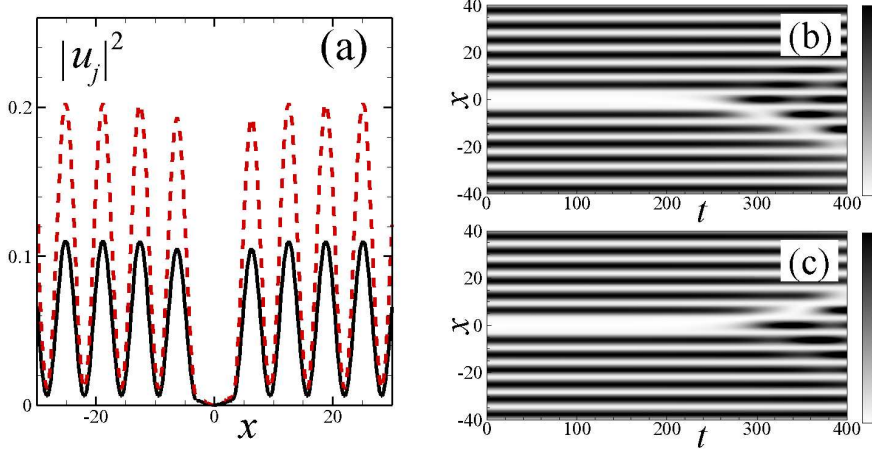
where “+” and “−” correspond to the first and second components, respectively,  $\epsilon$  will be taken as a small number and  $q$  is the perturbation wavenumber.

For the analysis we will fix the nonlinearity for the first component  $g_{11} = 1$  and change the nonlinearity for the second component in the interval  $0 < g_{22} < 1$  and the intercomponent interaction  $g_{12} = g_{21} = g$  in the interval  $-\sqrt{g_{22}} < g < g_{22}$ . Our results show that for small  $\alpha \ll 1$  the dark-dark solution is dynamically stable in the full range  $0 < g < g_{22}$  (see Fig. 1). However, in the interval  $-\sqrt{g_{22}} < g < 0$  the dark-dark solutions are unstable (see Fig. 2).

Also we have checked that by increasing the amplitude  $\alpha$  of the nonlinear inhomogeneity we found the instabilities take longer times to develop. As one can see in Fig. 3 for  $\alpha = 0.9$  the instability occurs near  $t \approx 250$  while in the previous example for  $\alpha = 0.3$  shown in Fig. 2 the solution breaks around  $t \approx 90$ .

### D. Bright-bright soliton solutions

Let us now consider the case  $\lambda < 0$  defined by Eq. (19b). In order to simplify the analysis we restrict ourselves to two particular choices of the constants  $\omega$ ,  $C_1$ ,  $C_2$  and  $C_3$ . The first one is  $\omega = 1$ ,  $C_1 = C_2 = 1/2$  and  $C_3 = 0$  in

FIG. 2: Same as in Fig. 1 but for  $g = -0.1$ .FIG. 3: Same as in Fig.1 but for  $\alpha = 0.9$ .

Eq. (19b). Thus  $c(x) = \cosh x$  and  $\lambda = -1/4$ ,  $E = E_1 = E_2 = 1/4$ , being  $\cos X(x) = -\tanh x$ , and the nonlinear coefficients are

$$g_{jk} = \frac{\tilde{g}_{jk}}{\cosh^3(x)} \quad (24)$$

Then  $0 \leq X \leq \pi$ , and to meet the boundary conditions  $u_j(\pm\infty) = 0$ , one has to impose  $U_j(0) = U_j(\pi) = 0$ . This means that the original infinite domain in Eqs. (2) is mapped into a bounded domain for Eqs. (13). By choosing the nonlinear coefficients to satisfy Eq. (15) and  $\sigma = -1$  we can use the solution (18c) to construct vector soliton solutions of the form

$$U_j(X) = \theta_j \mu k \sqrt{2(1-k^2)} \frac{\text{sn}(\mu X, k)}{\text{dn}(\mu X, k)}, \quad (25)$$

where  $\mu = \frac{1}{2} \sqrt{\frac{1}{1-2k^2}}$ .

The functions  $U_1(X)$  and  $U_2(X)$  satisfy  $U_1(0) = U_2(0) = 0$  and in order to meet  $U_1(\pi) = U_2(\pi) = 0$ , the condition  $\mu\pi = 2nK(k)$  where  $K(k)$  is the elliptic integral  $K(k) = \int_0^\pi (1 - k^2 \sin^2 \varphi)^{-1/2} d\varphi$  must hold. Thus, to satisfy the boundary conditions,  $k$  must be chosen to solve the following equation

$$4nK(k) \sqrt{1-2k^2} = \pi \quad \text{for } n = 1, 2, 3, \dots \quad (26)$$

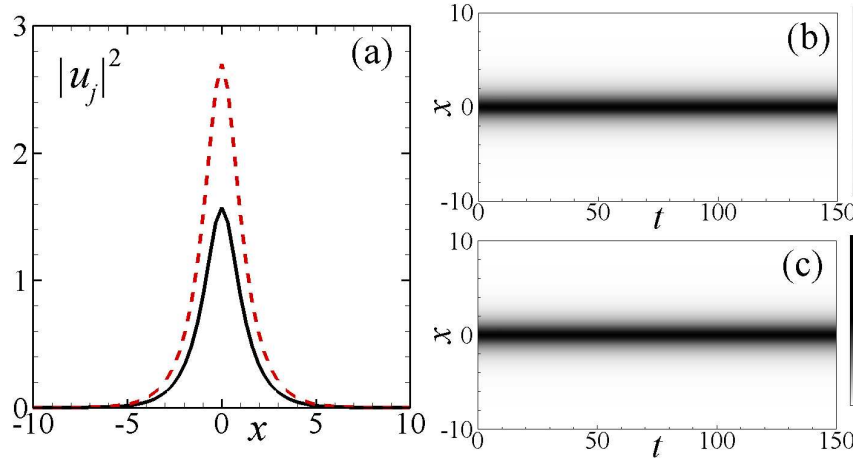


FIG. 4: Dynamics of a bright-bright soliton solution (27) with  $n = 1$  under a perturbation of the form (23). Shown are (a) the initial density profiles of the first (black solid) and second (red dashed) component. (b,c) Density plots of the time evolution of (b)  $|u_1(x,t)|^2$  (c)  $|u_2(x,t)|^2$ . Parameter values are  $g = 0.2$ ,  $g_{11} = -1$ ,  $g_{22} = -0.5$ ,  $\epsilon = 0.001$ , and  $q = 0.2$ .

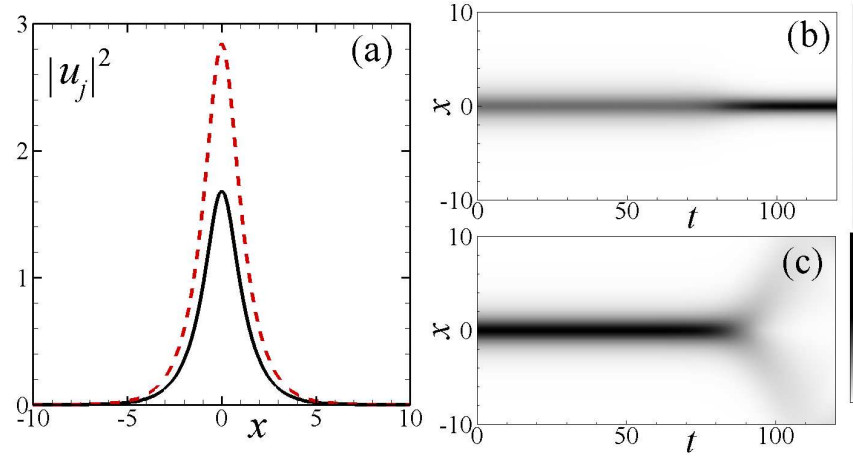


FIG. 5: Same as in Fig.4 but for  $g = 0.23$ .

It can be shown that for every integer number  $n$  this algebraic equation has only a solution  $k_n$  which means that there are an infinite number of solutions of Eqs. (2) of the form given by Eqs. (25). Moreover, each of those solutions has exactly  $n - 1$  zeroes. Then, using (11) we find a solution of Eqs. (2) of the form

$$u_j(x) = c(x)^{1/2} U_j(X(x)). \quad (27)$$

### E. Stability of the bright-bright solution (27)

Next we investigate the stability of the bright-bright solutions given by Eq. (27) by fixing  $g_{11} = -1$  and choosing different combinations of  $(g_{22}, g)$  (as in the previous case we consider  $g_{12} = g_{21} = g$ ). First we start with the case  $n = 1$  in Eq. (26). In this case for each value of  $g_{22}$  we find that there exists a critical value of  $g = g_{cr}$  such that for  $g < g_{cr}$  the vector solution is stable and for  $g > g_{cr}$  the solution becomes modulationally unstable. As examples Figs. 4 and 5 show that the critical value  $g_{cr}$  lies between  $g = 0.2$  and  $g = 0.23$ . It should be stressed here that in both regions of existence of the solutions, i.e.  $g < g_{cr}$  and  $g > g_{cr}$ , they are linearly stable, i.e. there are no negative or complex eigenvalues  $\Omega_j$ , ( $j = 1, 2$ ). However dynamically, the solutions for  $g > g_{cr}$  suffer a phase “separation” where one of the component extrudes the other component, a phenomenon also observed in related systems Ref. [51].

As to solutions with  $n = 2$  and 3, the linear stability analysis shows that all high-order solutions are linearly unstable.



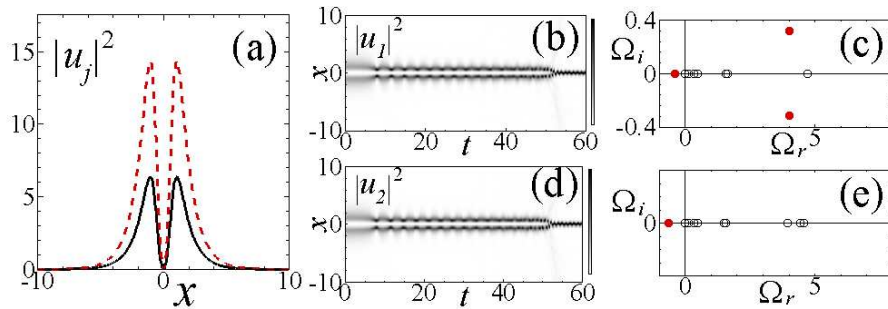


FIG. 6: Dynamics of a  $n = 2$  bright-bright solution given by Eq. (27) under a perturbation of the form (23). Shown are (a) the initial density profiles of the first (black solid) and second (red dashed) components. (b,d) Density plot of the time evolution of (b)  $|u_1(x,t)|^2$  (d)  $|u_2(x,t)|^2$ . (c,e) Part of the spectrum of the eigenvalues  $\Omega_i(\Omega_r)$  for the first (c) and second (e) components, calculated from Eq. (A5). The red dots on the spectrum show the existence of negative and complex eigenvalues. Parameter values are  $g = -0.1$ ,  $g_{11} = -1$ ,  $g_{22} = -0.5$ ,  $\epsilon = 0.001$ ,  $q = 0.2$ .

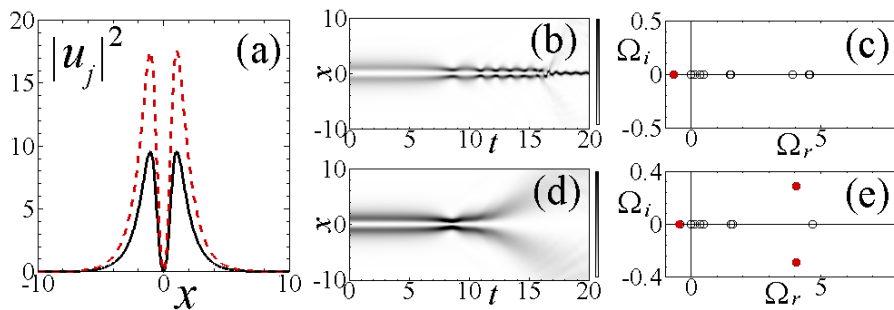


FIG. 7: Same as in Fig. 6 but for  $g = 0.1$ .

In the case  $n = 2$ , the linearized equations (see Appendix A) have one pure negative eigenvalue  $\Omega_r = \text{Re}(\Omega) < 0$  for both components and also two complex conjugated eigenvalues with non-zero imaginary part  $\Omega_i = \text{Im}(\Omega) \neq 0$  for the first component only (see panels (c), (e) in Figs.6 and 7).

The numerical integration of Eqs. (1) with perturbed initial bright-bright solution with  $n = 2$  confirms the results of the linear stability analysis. As we can see in panels (b) and (d) of Figs. 6 and 7, the dynamics becomes unstable for small times and consists of oscillatory behavior at the initial stages of instability. When the instability develops fully the initial profile of the solution (having two peaks) degenerates into a one-peak stationary profile corresponding to bright-bright solution with  $n = 1$  (Fig.6). In the case of  $g > 0$  due to the instability, the second component splits into two wavepackets which move outwards the center while the first component transforms into one-peak bright soliton solution (see Fig.7), that is a stable mode of the system.

In the case of  $n = 3$  one more pair of complex conjugated eigenvalues appears in comparison with the case  $n = 2$  what corresponds to the more pronounced instability which starts to develop earlier then in previous case and in the case  $g < 0$  consists of transformation of the three-peaks profile of the solution into one-peak oscillating solution (see Figs.8 and 9) while in the case  $g > 0$  the perturbed bright-bright solution disappears after strong radiation emission-

#### IV. SYSTEMS WITH QUADRATIC POTENTIALS

##### A. Construction of families of solutions

In this section, we construct explicit solutions of Eqs. (2) when either (or both) of the external potentials  $V_1$  or  $V_2$  are different from zero. We will consider the case when  $\lambda_1 = \lambda_2 = \lambda$  and  $E_1 = E_2 = 0$ . By choosing  $c(x) = e^{-\lambda x^2}$  and substituting into Eqs. (14) and (14b) we get quadratic trapping potentials of the form  $V_1(x) = V_2(x) = \lambda^2 x^2$ . These Gaussian nonlinearities can be generated by controlling the Feshbach resonances optically using a Gaussian beam (see



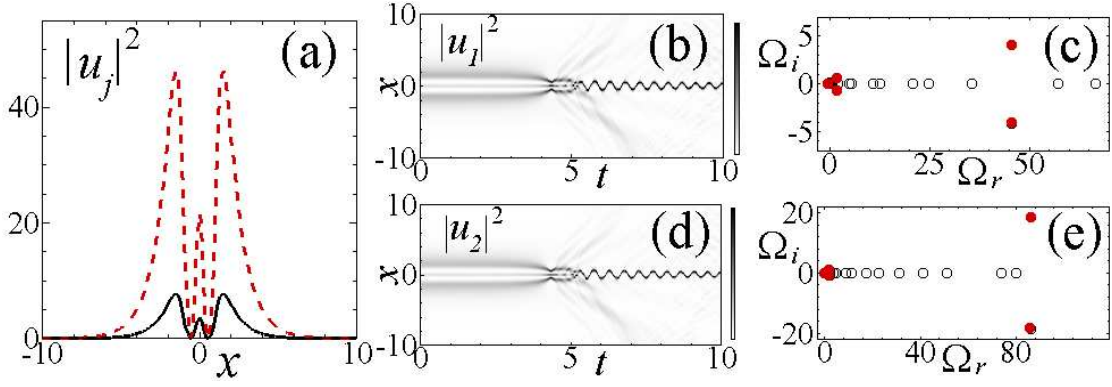


FIG. 8: Dynamics of a bright-bright solution given by Eq. (27) with  $n = 3$  under a perturbation of the form (23). Shown are (a) the initial density profiles of the first (black solid) and second (red dashed) component. (b,d) Density plot of the time evolution of (b)  $|u_1(x, t)|^2$  (d)  $|u_2(x, t)|^2$ . (c,e) Part of the spectrum of the eigenvalues  $\Omega_i(\Omega_r)$  for the first (c) and second (e) components, calculated from Eq. (A5). The red dots on the spectrum show the existence of negative and complex eigenvalues. Parameter values are  $g = -0.4$ ,  $g_{11} = -1$ , and  $g_{22} = -0.5$ ,  $\epsilon = 0.001$ ,  $q = 0.2$ .

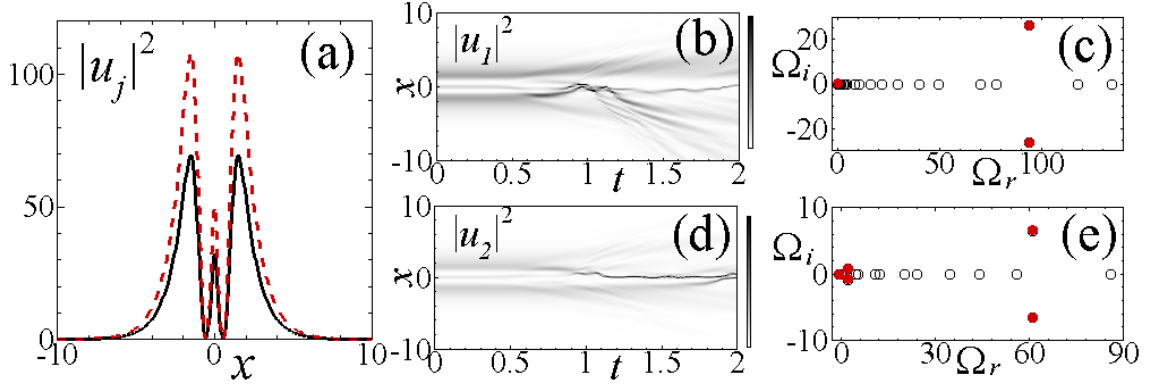


FIG. 9: Same as in Fig. 8 but for  $n = 3$  and  $g = 0.4$ .

e.g. [52]). Here we also take  $K = 0$ , thus

$$g_{jk}(x) = \tilde{g}_{jk} \exp(3\lambda x^2), \quad j, k = 1, 2. \quad (28)$$

Our canonical transformation is given by  $X(x) = \int_0^x ds \exp(\lambda s^2) = \frac{1}{2}\sqrt{-\frac{\pi}{\lambda}} \operatorname{erf} \sqrt{-\lambda} x$ . This leads us to a restriction on the sign of the chemical potential  $\lambda$  which must be less than zero,  $\lambda < 0$ . In this case Eq. (13) becomes

$$-\frac{d^2 U_1}{dX^2} + \tilde{g}_{11} U_1^3 + \tilde{g}_{12} U_1 U_2^2 = 0, \quad (29a)$$

$$-\frac{d^2 U_2}{dX^2} + \tilde{g}_{22} U_2^3 + \tilde{g}_{21} U_1^2 U_2 = 0. \quad (29b)$$

Note that the range of  $X$  is again finite since  $-\frac{1}{2}\sqrt{-\frac{\pi}{\lambda}} \leq X \leq \frac{1}{2}\sqrt{-\frac{\pi}{\lambda}}$ , and hence, we can again construct many localized solutions to Eq. (2) starting from solutions of Eqs. (29) which satisfy the boundary conditions  $U_j(\pm \frac{1}{2}\sqrt{-\frac{\pi}{\lambda}}) = 0$ ,  $j = 1, 2$ .

Using the condition (15) we can reduce Eqs. (29) to

$$-\frac{d^2 \Phi}{dX^2} + \sigma \Phi^3 = 0, \quad (30)$$

where we introduce new field  $\Phi$  which is related with original fields through the relation (17) and  $\sigma = \operatorname{sgn}[(\tilde{g}_{11}\tilde{g}_{22} -$

$\tilde{g}_{12}\tilde{g}_{21})(\tilde{g}_{jj} - \tilde{g}_{12})]$ . For  $\sigma = -1$  this equation has two solutions

$$\Phi^{(1)}(X) = \mu \operatorname{cn}(\mu X, k_*), \quad (31a)$$

$$\Phi^{(2)}(X) = \frac{\mu}{\sqrt{2}} \frac{\operatorname{sn}(\mu X, k_*)}{\operatorname{dn}(\mu X, k_*)}, \quad (31b)$$

with  $k_* = 1/\sqrt{2}$ , which solve the system (29) and that  $\Phi^{(1)}(X)$  and  $\Phi^{(2)}(X)$  vanish when  $\mu X = (2n+1)K(k_*)$  and  $\mu X = 2nK(k_*)$  correspondingly. Thus we come to an infinite number of solutions of system (29) under zero boundary conditions on the new finite interval, which correspond to different values of  $\mu$ .

Finally, localized solutions of the NLS system (2), are given by

$$u_{jn}(x) = 2n\sqrt{\frac{-\lambda}{\pi}} K(k_*) \theta_j e^{x^2/2} \operatorname{cn}(\chi_n(x), k_*), \quad n = 1, 3, \dots \quad (32)$$

and

$$u_{jn}(x) = \sqrt{\frac{-2\lambda}{\pi}} n K(k_*) \theta_j e^{x^2/2} \frac{\operatorname{sn}(\chi_n(x), k_*)}{\operatorname{dn}(\chi_n(x), k_*)}, \quad n = 2, 4, \dots \quad (33)$$

with

$$\chi_n(x) = nK(k_*) \operatorname{erf} \sqrt{-\lambda} x. \quad (34)$$

It can be shown by simple asymptotic analysis that the last factors in Eqs. (32) and (33) tend to zero as  $x \rightarrow \pm\infty$  faster than  $\exp(-x^2/2)$  and that these are indeed *localized* solutions of our problem.

## B. Dynamics and stability of the solutions

Let us now check the stability of the bright-bright soliton solutions in the presence of the parabolic potential. First we consider solutions given by Eq. (32) with  $n = 1$ . As in the case of the bright-bright solution without external potential from Section III D here solution for  $n = 1$  is also linearly stable in all the domain of existence of the solution. However, by checking the dynamical stability under small perturbation of the initial exact solution we found that for fixed values of intra-species interactions  $g_{11} = -1$  and  $g_{22} = -0.5$  there is the domain of stability  $\sqrt{g_{11}g_{22}} < g \leq 0$  (see an example in Fig. 10) and also that there is a domain where the solution is unstable  $g > 0$  (an example in Fig. 11). As it is clear from Fig. 11(b), (c) there is a phase separation effect related to the immiscibility of the two-component mixture. In this case the first component is shifted to negative  $x$  values while the second component is displaced to positive  $x$  values; however this separation rather small and components continue to have a significant overlapping.

Now let us consider stability of the bright-bright solution for  $n = 2$ . In contrast to the case without linear potential studied previously, here the solution is linearly stable in all of its domain of existence. From the dynamical stability analysis we find that, in the domain  $\sqrt{g_{11}g_{22}} < g \leq 0$ , the solution is dynamically stable while for  $g > 0$  as in the previous case of the  $n = 1$  solution becomes unstable due to the immiscibility condition between the two components. Examples are shown in Fig. 12 that has a stable dynamics and in Fig. 13 where unstable behaviour of the bright-bright solution for  $n = 2$  is observed. In contrast to the case of bright-bright solution without external potential in the case at hand the second component due to reflection from the parabolic potential starts to oscillate.

Finally, when moving to solitons with  $n = 3$  we observe that in the full domain of existence of the bright-bright solution, it is linearly as well as dynamically unstable (see Figs. 14 and 15).

## V. DARK-BRIGHT SOLITON SOLUTIONS

### A. Dark-bright soliton solutions

Up to now we have focused our attention in constructing vector extensions of the solutions known for the scalar case with spatially dependent nonlinear coefficients and studying their stability. In this section we will consider nontrivial solutions where the functional form of both components is completely different.

Dark-bright soliton solutions have been extensively studied in the context of BEC after the pioneering work of Busch and Anglin [45], where the coefficients  $\tilde{g}_{ij} = 1$ ,  $j, k = 1, 2$ , what corresponds to the well-known integrable Manakov system [41].

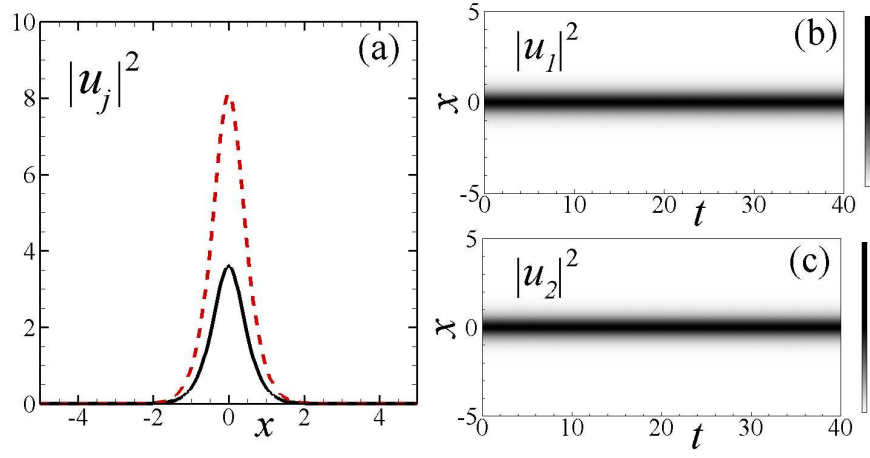


FIG. 10: Dynamics of a  $n = 1$  bright-bright soliton solution (32) in a parabolic potential  $V_j(x) = \lambda^2 x^2$ , ( $j = 1, 2$ ) under a perturbation of the form (23). Shown are (a) the initial density profiles of the first (black solid) and second (red dashed) component. (b,c) Density plot of the time evolution of (b)  $|u_1(x, t)|^2$  (c)  $|u_2(x, t)|^2$ . Parameter values are  $g = -0.1$ ,  $g_{11} = -1$ ,  $g_{22} = -0.5$ ,  $\lambda = -1$ ,  $\epsilon = 0.001$ ,  $q = 0.2$ .

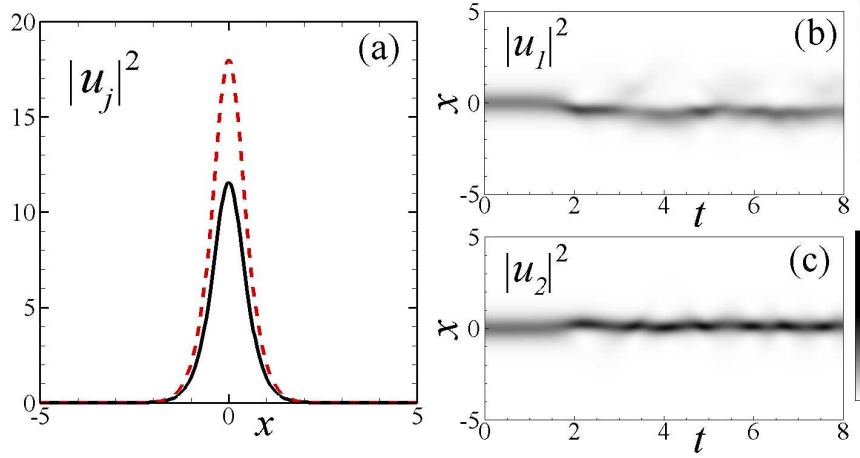


FIG. 11: Same as in Fig. 10 but for  $g = 0.4$ .

As in the dark-dark case to construct dark-bright solution it is natural to consider  $V_1(x) = 0$  for the dark component,  $\lambda_1 = \lambda_2 = \lambda = \omega^2/4 > 0$  and correspondingly  $c(x) = 1 + \alpha \cos \omega x$ . From Eq. (13) we get the relation

$$E_1 = \lambda(1 - \alpha^2), \quad (35a)$$

$$E_2 = E_1 - V_2(x)c(x)^2, \quad (35b)$$

and taking the potential  $V_2(x)$  of the form

$$V_2(x) = K(1 + \alpha \cos(\omega x))^{-2}, \quad (36)$$

from (35a) we get that  $E_1 \neq E_2$ , and

$$K = E_1 - E_2. \quad (37)$$

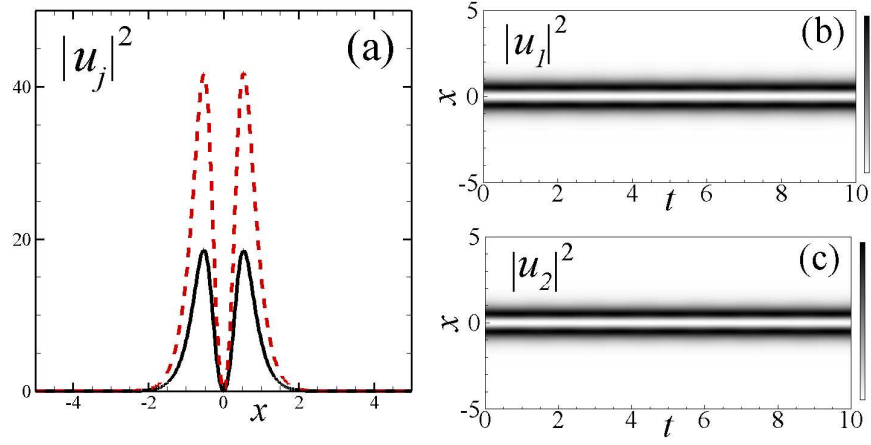


FIG. 12: Dynamics of a  $n = 2$  bright-bright soliton solution (33) in a parabolic potential  $V_j(x) = \lambda^2 x^2$ , ( $j = 1, 2$ ) under a perturbation of the form (23). Shown are (a) the initial density profiles of the first (black solid) and second (red dashed) component. (b,c) Density plot of the time evolution of (b)  $|u_1(x, t)|^2$  (c)  $|u_2(x, t)|^2$ . Parameter values are  $g = -0.1$ ,  $g_{11} = -1$ ,  $g_{22} = -0.5$ ,  $\lambda = -1$ ,  $\epsilon = 0.001$ ,  $q = 0.2$ .

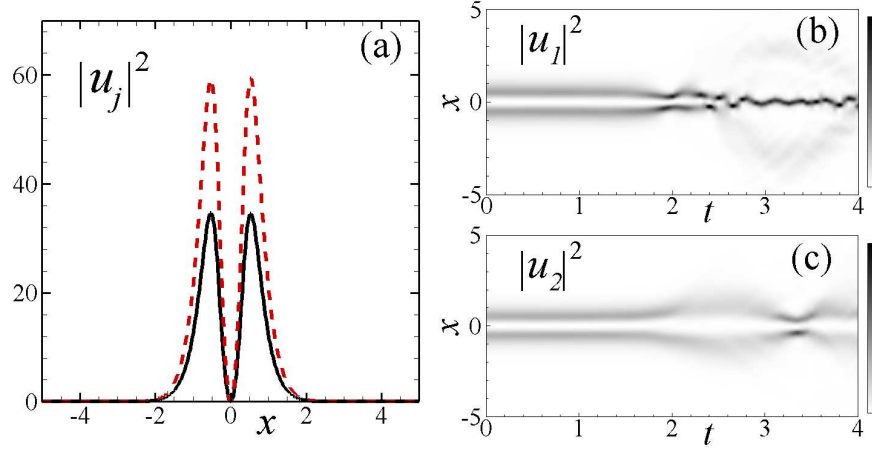


FIG. 13: Same as in Fig. 10 but for  $g = 0.2$ .

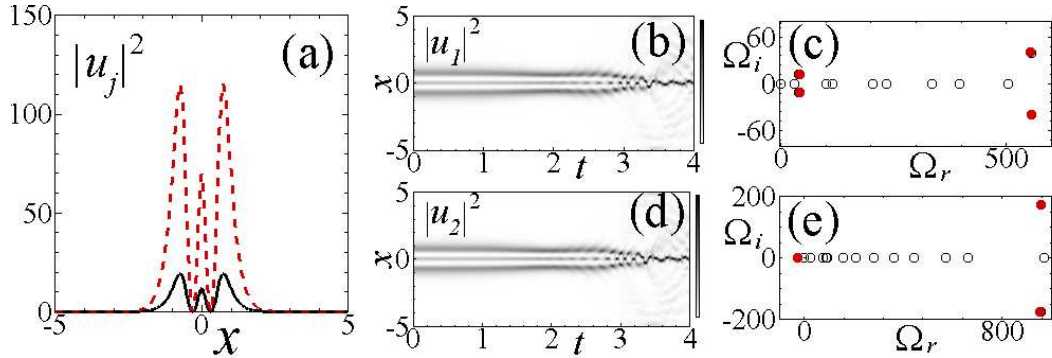
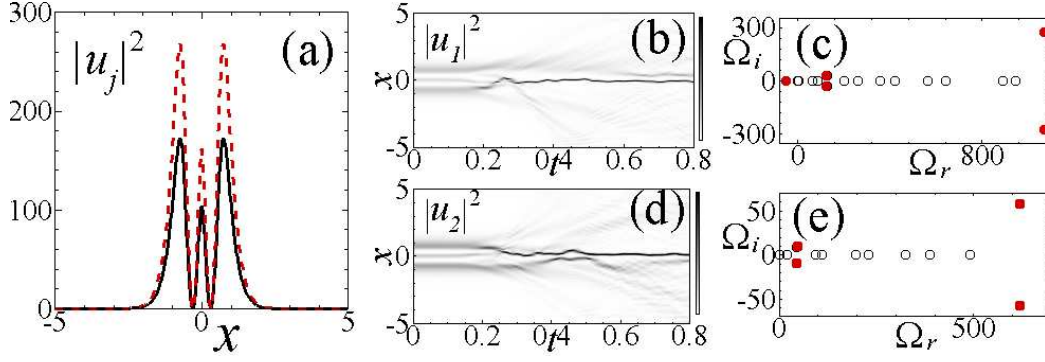


FIG. 14: Dynamics of a  $n = 3$  bright-bright soliton solution (32) in a parabolic potential  $V_j(x) = \lambda^2 x^2$ , ( $j = 1, 2$ ) under a perturbation of the form (23). Shown are (a) the initial density profiles of the first (black solid) and second (red dashed) component. (b,c) Density plot of the time evolution of (b)  $|u_1(x, t)|^2$  (c)  $|u_2(x, t)|^2$ . Parameter values are  $g = -0.4$ ,  $g_{11} = -1$ ,  $g_{22} = -0.5$ ,  $\lambda = -1$ ,  $\epsilon = 0.001$ ,  $q = 0.2$ .

FIG. 15: Same as in Fig.14 but for  $g = 0.4$ .

Therefore, we can choose as solutions of Eqs. (13) the following dark-bright soliton solutions:

$$U_1(X) = \sqrt{\frac{E_1}{\tilde{g}_{11}}} \tanh(\mu X), \quad (38a)$$

$$U_2(X) = \sqrt{\frac{E_1}{\tilde{g}_{11}} \frac{(\tilde{g}_{21} - \tilde{g}_{11})}{(\tilde{g}_{22} - \tilde{g}_{12})}} \operatorname{sech}(\mu X), \quad (38b)$$

where

$$\mu = \sqrt{\frac{E_1}{2\tilde{g}_{11}} \frac{(\tilde{g}_{11}\tilde{g}_{22} - \tilde{g}_{12}\tilde{g}_{21})}{(\tilde{g}_{22} - \tilde{g}_{12})}}, \quad (39a)$$

$$E_2 = \frac{\tilde{g}_{21}}{\tilde{g}_{11}} E_1 - \mu^2. \quad (39b)$$

As it is clear in order to have real solutions one has to satisfy one of the following sets of conditions for the nonlinear coefficients: either  $\tilde{g}_{22} > \tilde{g}_{12}$ ,  $\tilde{g}_{21} > \tilde{g}_{11}$ ,  $\tilde{g}_{11}\tilde{g}_{22} > \tilde{g}_{12}\tilde{g}_{21}$  or  $\tilde{g}_{22} < \tilde{g}_{12}$ ,  $\tilde{g}_{21} < \tilde{g}_{11}$ ,  $\tilde{g}_{11}\tilde{g}_{22} < \tilde{g}_{12}\tilde{g}_{21}$ .

Then, the dark-bright type solution of Eq. (2) is given by

$$u_1(x) = (1 + \alpha \cos(\omega x))^{1/2} U_1(X(x)), \quad (40a)$$

$$u_2(x) = (1 + \alpha \cos(\omega x))^{1/2} U_2(X(x)), \quad (40b)$$

where  $X(x)$  is given by Eq. (21).

### B. Stability of dark-bright soliton solutions

Dark-bright solutions given by Eqs. (40) are remarkably robust to perturbations as one would in principle expect. All of our simulations of perturbed dark-bright solitons have lead to stable behavior.

A couple of examples are shown in Figs. 16 and 17 for two different values of the nonlinearity  $g_{11} = 0.26$  (Fig. 16) and  $g_{11} = 0.49$  (Fig. 17).

## VI. CONCLUSIONS

In conclusion, we have used Lie symmetries and canonical transformations to construct explicit solutions of coupled nonlinear Schrödinger systems with spatially inhomogeneous nonlinearities starting from those of spatially homogeneous coupled nonlinear Schrödinger equations.

Although we have restricted our attention to a few selected examples of physical relevance, the range of nonlinearities and potentials for which this can be done is very wide. We have constructed explicit solutions for  $V = 0$  with localized and periodic nonlinearities and discussed the parameter regimes for which solutions are stable. A similar methodology

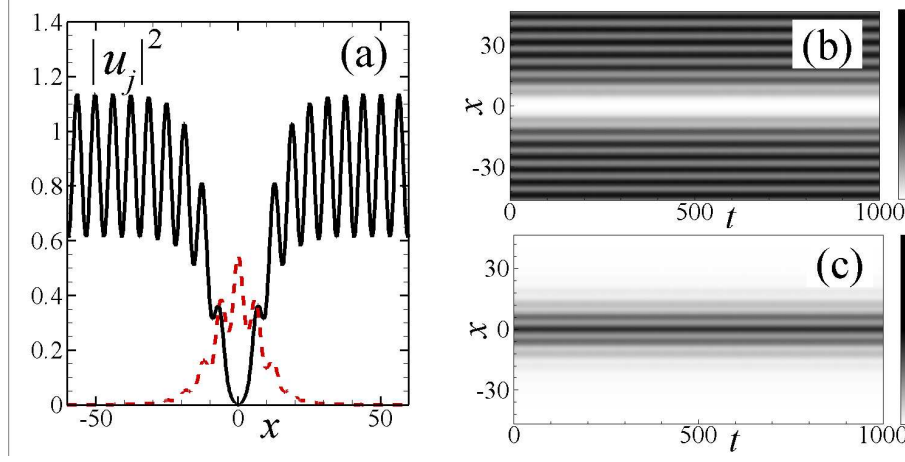


FIG. 16: Dynamics of a dark-bright soliton solution given by Eq. (40) under a perturbation of the form (23). Shown are (a) the initial density profiles of the dark (black solid) and bright (red dashed) components, and ensity plots of the time evolution of (b)  $|u_1(x,t)|^2$  (c)  $|u_2(x,t)|^2$ . Parameter values are  $g_{11} = 0.26$ ,  $g_{22} = 1$ ,  $g = 0.5$ ,  $\alpha = 0.3$ ,  $\omega = 1$ ,  $\epsilon = 0.02$ ,  $q = 0.2$ .

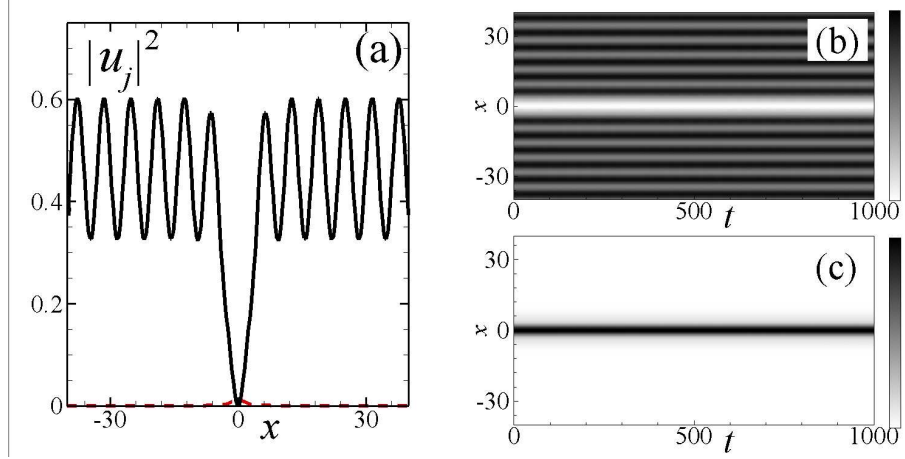


FIG. 17: Same as in Fig. 16 but for  $g_{11} = 0.49$ .

has been applied to construct and study the stability of specific families of solutions when the potential is quadratic on the space coordinates. Finally we have constructed dark-bright soliton solutions that are not direct extensions of known solutions of the scalar spatially inhomogeneous case.

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### Appendix A: Linear stability analysis

Let us look for perturbed solutions of Eq. (1)

$$u_j(x,t) = (u_{0j} + A_j(x,t)) e^{i\lambda_j t}, \quad (\text{A1})$$

where  $u_{0j}$  are real solutions of the stationary problem (2) and  $|A_j| \ll u_{0j}$  are perturbations. Substituting (A1) in Eq. (2) and linearizing it with respect to  $A_j(x)$  we obtain

$$i\dot{A}_j = (\lambda_j + V(x)) A_j - \nabla^2 A_j + (g_{jj}|u_{0j}|^2 + g|u_{0(3-j)}|^2) A_j + g_{jj}|u_{0j}|^2 \bar{A}_j + 2gu_{0j}u_{0(3-j)} (A_{3-j} + \bar{A}_{3-j}) \quad (\text{A2})$$

Decomposing  $A_j$  into its real and imaginary parts,  $A_j(x, t) = P_j(x, t) + iQ_j(x, t)$ , defining the operators

$$L_j^{(+)} = \lambda_j - \nabla^2 + 3g_{jj} (u_{0j})^2 + g (u_{0(3-j)})^2, \quad (\text{A3a})$$

$$L_j^{(-)} = \lambda_j - \nabla^2 + g_{jj} (u_{0j})^2 + g (u_{0(3-j)})^2, \quad (\text{A3b})$$

and introducing the notation  $S(x) = 2gu_{01}(x)u_{02}(x)$ , we obtain

$$\begin{pmatrix} \dot{P}_j \\ -\dot{Q}_j \end{pmatrix} = \begin{pmatrix} 0 & L_j^{(-)} \\ L_j^{(+)} & 0 \end{pmatrix} \begin{pmatrix} P_j \\ Q_j \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ S(x) & 0 \end{pmatrix} \begin{pmatrix} P_{3-j} \\ Q_{3-j} \end{pmatrix}. \quad (\text{A4})$$

Looking for solutions in the form  $P_j(x, t) = p_j(x)e^{wt}$  and  $Q_j(x, t) = q_j(x)e^{wt}$ , and defining  $\Omega = -w^2$ , Eqs. (A4) reduce to

$$\Omega \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} L_1^{(-)}L_1^{(+)} & L_1^{(-)}S(x) \\ L_2^{(-)}S(x) & L_2^{(-)}L_2^{(+)} \end{pmatrix} \begin{pmatrix} p_1(x) \\ p_2(x) \end{pmatrix}. \quad (\text{A5})$$

Thus, a solitary wave solution  $u_{0,j}$  will be linearly unstable when either  $\Omega_r = \text{Re}(\Omega) < 0$  or  $\Omega_i = \text{Im}(\Omega) \neq 0$ .

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- [1] C. Sulem and P. Sulem, *The Nonlinear Schrödinger Equation: Self-focusing and Wave Collapse* (Springer, Berlin, 2000).
  - [2] L. Vázquez, L. Streit, V. M. Pérez-García, Eds., *Nonlinear Klein-Gordon and Schrödinger systems: Theory and Applications*, World Scientific, Singapur (1997).
  - [3] R. Carretero-González, D. J. Frantzeskakis and P. G. Kevrekidis, Nonlinear waves in Bose-Einstein condensates: physical relevance and mathematical techniques, *Nonlinearity* **21**, R139 (2008).
  - [4] V. M. Pérez-García, N. G. Berloff, P. G. Kevrekidis, V. V. Konotop, B. A. Malomed, Nonlinear phenomena in degenerate quantum gases, *Physica D*, **238**, 1289 (2009).
  - [5] H. Saito, and M. Ueda, Dynamically Stabilized Bright Solitons in a Two-Dimensional Bose-Einstein Condensate, *Phys. Rev. Lett.* **90**, 040403 (2003).
  - [6] F. Kh. Abdullaev, J. G. Caputo, B. A. Malomed, and R. A. Kraenkel, Controlling collapse in Bose-Einstein condensates by temporal modulation of the scattering length, *Phys. Rev. A* **67**, 013605 (2003).
  - [7] P.G. Kevrekidis, G. Theocharis, D.J. Frantzeskakis and B.A. Malomed, Feshbach Resonance Management for Bose-Einstein Condensates, *Phys. Rev. Lett.* **90**, 230401 (2003).
  - [8] F. Kh. Abdullaev, A.M. Kamchatnov, V. V. Konotop, and V. A. Brazhnyi, Adiabatic Dynamics of Periodic Waves in Bose-Einstein Condensates with Time Dependent Atomic Scattering Length, *Phys. Rev. Lett.* **90**, 230402 (2003).
  - [9] G. D. Montesinos, V. M. Pérez-García, and P. Torres, Stabilization of solitons of the multidimensional nonlinear Schrödinger equation: matter-wave breathers, *Physica D* **191**, 193 (2004).
  - [10] S. K. Adhikari, Stabilization of bright solitons and vortex solitons in a trapless three-dimensional Bose-Einstein condensate by temporal modulation of the scattering length, *Phys. Rev. A* **69**, 063613 (2004).
  - [11] G. D. Montesinos, V. M. Pérez-García, and H. Michinel, Stabilized Two-Dimensional Vector Solitons, *Phys. Rev. Lett.* **92**, 133901 (2004).
  - [12] V. A. Brazhnyi and V. V. Konotop, Management of matter waves in optical lattices by means of the Feshbach resonance, *Phys. Rev. A* **72**, 033615 (2005).
  - [13] V. V. Konotop and P. Pacciani, Collapse of Solutions of the Nonlinear Schrödinger Equation with a Time-Dependent Nonlinearity: Application to Bose-Einstein Condensates, *Phys. Rev. Lett.* **94**, 240405 (2005).
  - [14] P. G. Kevrekidis, D. E. Pelinovsky, and A. Stefanov, Nonlinearity management in higher dimensions, *J. Phys. A* **39**, 479 (2006).
  - [15] A. Itin, T. Morishita, and S. Watanabe, Reexamination of dynamical stabilization of matter-wave solitons, *Phys. Rev. A* **74**, 033613 (2006).
  - [16] M. I. Rodas-Verde, H. Michinel and V. M. Pérez-García, Controllable soliton emission from a Bose-Einstein condensate, *Phys. Rev. Lett.* **95**, 153903 (2005).
  - [17] F. K. Abdullaev and J. Garnier, Propagation of matter-wave solitons in periodic and random nonlinear potentials, *Phys. Rev. A* **72**, 061605(R) (2005).



- [18] G. Theocharis, P. Schmelcher, P. G. Kevrekidis, and D. J. Frantzeskakis, Matter-wave solitons of collisionally inhomogeneous condensates, *Phys. Rev. A* **72**, 033614 (2005).
- [19] H. Sakaguchi and B. A. Malomed, Matter-wave solitons in nonlinear optical lattices, *Phys. Rev. E* **72**, 046610 (2005).
- [20] H. Sakaguchi and B. A. Malomed, Two-dimensional solitons in the Gross-Pitaevskii equation with spatially modulated nonlinearity, *Phys. Rev. E* **73**, 026601 (2006).
- [21] J. Garnier and F. K. Abdullaev, Transmission of matter-wave solitons through nonlinear traps and barriers, *Phys. Rev. A* **74**, 013604 (2006).
- [22] A. V. Carpentier, H. Michinel, M. I. Rodas-Verde, and V. M. Pérez-García, Analysis of an atom soliton laser based on the spatial control of the scattering length, *Phys. Rev. A*, **74**, 013619 (2006).
- [23] Y. V. Bludov, and V. V. Konotop, Localized modes in arrays of boson-fermion mixtures, *Phys. Rev. A*, **74**, 043616 (2006).
- [24] J. Belmonte-Beitia, V. M. Pérez-García, V. Vekslerchik, and P. J. Torres, Lie symmetries and solitons in nonlinear systems with spatially inhomogeneous nonlinearities, *Phys. Rev. Lett.* **98** 064102 (2007).
- [25] X-Y Tang and P. K. Shukla, Solution of the one-dimensional spatially inhomogeneous cubic-quintic nonlinear Schrödinger equation with an external potential, *Phys. Rev. A* **76**, 013612, (2007)
- [26] P. Niarchou, G. Theocharis, P. G. Kevrekidis, P. Schmelcher, and D. J. Frantzeskakis, Soliton oscillations in collisionally inhomogeneous attractive Bose-Einstein condensates, *Phys. Rev. A* **76**, 023615 (2007).
- [27] P. Torres, Guided waves in a multi-layered optical structure, *Nonlinearity* **19**, 2103 (2006).
- [28] D. A. Zezyulin, G. L. Alfimov, V. V. Konotop and V. M. Pérez-García, Control of nonlinear modes by scattering-length management in Bose-Einstein condensates, *Phys. Rev. A* **76**, 013621 (2007).
- [29] Y. Sivan, G. Fibich, and M. I. Weinstein, Waves in nonlinear lattices: Ultrashort optical pulses and Bose-Einstein condensates, *Phys. Rev. Lett.* **97**, 193902 (2006).
- [30] M. A. Porter, P. G. Kevrekidis, B. A. Malomed, D. J. Frantzeskakis, Modulated amplitude waves in collisionally inhomogeneous Bose-Einstein condensates, *Physica D*, **229**, 104-115 (2007).
- [31] J. Belmonte-Beitia, V. M. Pérez-García, V. Vekslerchik and P. J. Torres, Lie Symmetries, qualitative analysis and exact solutions of Nonlinear Schrödinger equations with inhomogeneous nonlinearities, *Discr. and Cont. Dyn. Sys. B* **9**, 221 (2008).
- [32] J. Belmonte-Beitia, V. M. Pérez-García, V. Vekslerchik, V. V. Konotop, Localized Nonlinear Waves in Systems with Time- and Space-Modulated Nonlinearities, *Phys. Rev. Lett.* **100**, 164102 (2008).
- [33] A.S. Rodrigues, P.G. Kevrekidis, M.A. Porter, D.J. Frantzeskakis, P. Schmelcher and A.R. Bishop, Matter-wave solitons with a periodic, piecewise-constant scattering length, *Phys. Rev. A* **78**, 013611 (2008).
- [34] V. M. Pérez-García, and R. Pardo, Localization phenomena in nonlinear Schrödinger equations with spatially inhomogeneous nonlinearities: Theory and applications to Bose-Einstein condensates, *Physica D*, **238** 1352 (2009).
- [35] J. Belmonte-Beitia, V. M. Pérez-García, V. Vekslerchik, V. V. Konotop, Localized and periodic exact solutions to the nonlinear Schrodinger equation with spatially modulated parameters: Linear and nonlinear lattices, *Chaos, Solitons and Fractals*, **41**, 1158 (2009).
- [36] J. Belmonte-Beitia and J. Cuevas, Solitons for the cubic-quintic nonlinear Schrödinger equation with time- and space-modulated coefficient, *J. Phys. A: Math. Theor.* **42**, 165201 (2009).
- [37] Y. V. Kartashov, B. A. Malomed, V. A. Vysloukh, L. Torner, Vector solitons in nonlinear lattices, to appear in *Optics Letters*.
- [38] R. Y. Hao, R. C. Zang, L. Li, G. S. Zhou, Solutions for the propagation of light in nonlinear optical media with spatially inhomogeneous nonlinearities, *Opt. Commun.* **281**, 1256 (2007).
- [39] J. Belmonte-Beitia, Exact solutions for the quintic nonlinear Schrodinger equation with inhomogeneous nonlinearity, *Chaos, Solitons and Fractals*, **41**, 1005 (2009).
- [40] C. M. Khalique, A. Biswas, A Lie symmetry approach to nonlinear Schrodinger's equation with non-Kerr law nonlinearity, *Comm. Nonlinear Sci. and Numer. Simul.* **14**, 4033 (2009).
- [41] S. A. Manakov, On the theory of two-dimensional stationary self-focusing of electromagnetic waves, *Sov. Phys. JETP* **38** (1974) 248-253.
- [42] V. E. Zakharov and E. I. Schulman, To the integrability of the system of two coupled nonlinear Schrödinger equations, *Physica D*, **4** 270 (1982).
- [43] V. E. Zakharov and S. V. Manakov, Construction of multidimensional nonlinear integrable systems and their solutions, *Funk. Anal. Ego Prilozh.* **19** 11 (1985).
- [44] D. Kaup and B. Malomed, Soliton Trapping and Daughter Waves in the Manakov Model, *Phys. Rev. A* **48** 1 (1993).
- [45] Th. Busch and J. R. Anglin, Dark-Bright solitons in Inhomogeneous Bose-Einstein Condensates, *Phys. Rev. Lett.* **87** 010401 (2001).
- [46] V. M. Pérez-García and J. B. Beitia, Symbiotic solitons in heteronuclear multicomponent Bose-Einstein condensates, *Phys. Rev. A* **72**, 033620 (2005).
- [47] J. Belmonte-Beitia, V. M. Pérez-García and V. E. Vekslerchik, Modulational instability, solitons and periodic waves in models of quantum degenerate Boson-Fermion mixtures, *Chaos, Solitons and Fractals*, **32**, 1268-1277 (2007).
- [48] Y. Cheng, Effective potential of two coupled binary matter wave bright solitons with spatially modulated nonlinearity, *J. Phys. B: At. Mol. Opt. Phys.* **42**, 205005 (2009).
- [49] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Springer Verlag, New York (1989).
- [50] P. G. L. Leach, An exact invariant for a class of time-dependent anharmonic oscillators with cubic anharmonicity, *J. Math. Phys.*, **22**, 465 (1981).
- [51] H.A. Cruz, V.A. Brazhnyi and V.V. Konotop, Partial delocalization of two-component condensates in optical lattices, *J.*

- Phys. B: At. Mol. Opt. Phys. **41**, 035304 (2008).
- [52] M. Theis, G. Thalhammer, K. Winkler, M. Hellwig, G. Ruff, R. Grimm, and J. H. Denschlag, Tuning the Scattering Length with an Optically Induced Feshbach Resonance, Phys. Rev. Lett. **93**, 123001 (2004).